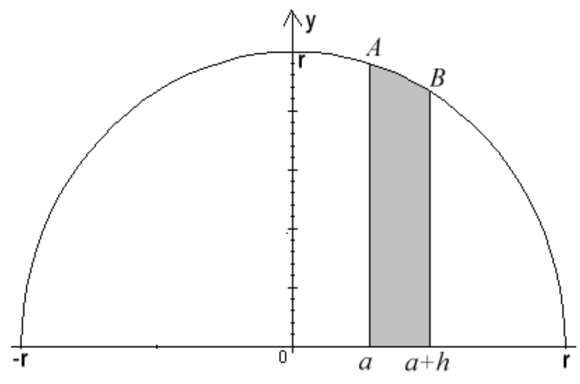
Math 1B Final Exam Problems – A sampling of fair game

1. Find the area above the *x*-axis and below the curve $y = \dfrac{1-x^2}{1+x^2}$.

2. Suppose the area in the first quadrant bound by $y = -x + \sin\left(\dfrac{\pi x}{2}\right)$ is revolved around the *y*-axis. Assume all variables represent length in meters.
a. Sketch a diagram showing the volume and sketch in a representative cylindrical shell.
b. Set an integral to compute the volume generated.
c. Evaluate the integral to find the volume generated. .

3. Find the length of the curve $5y^3 = x^2$ that lies inside the circle of radius 6 centered at the origin.

4. If a spherical loaf of bread is cut into slices of equal width, each slice will have the same amount of crust. To verify this, suppose that a semicircle $y = \sqrt{r^2 - x^2}$ is revolved about the *x*-axis to generate a sphere. Let *AB* be the arc of a semicircle that stands above an interval of length *h* on the *x*-axis. Show that the surface area swept out by *AB* does not depend on the location of the interval. That is, the length of *AB* in the diagram is independent of *a*.



5. A spherical tank or radius *r* is filled with water. Find the minimum work required to empty the tank through a hole at its top. Be clear to show how your tank is oriented in a coordinate system.

6. Show that the area of the region between the curve $y = \dfrac{1}{1+x^2}$ and the entire *x*-axis is the same as the area of the unit disk, $x^2 + y^2 \le 1$.

7. Find the first three non-zero terms in the Maclaurin series for $f(x) = \cos^3 x$
a. By cubing the Maclaurin series for $y = \cos x$
b. By using the identity $\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$ and adding terms.

8. Approximate the integral $\int_{-0.1}^{0.1} e^{-x^2}\,dx$ using
a. Simpson's rule with $n = 4$.

b. The Taylor polynomial of degree 6.

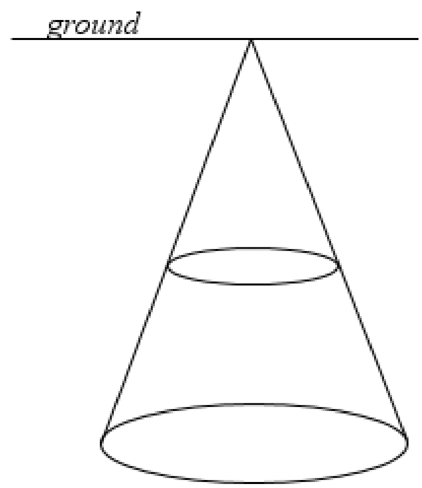
9. Use a Maclaurin series to approximate $\dfrac{3}{e}$. How many terms are necessary to approximate this to the nearest 5×10^{-13}?

10. Use integration by parts to find the area below $y = e^{-x} \cos \pi x$ between x = 0 and x = ½ .
Hint: To find an antiderivative, you'll need to do parts twice to see the integral recur.

11. Consider the area between the curves $y = \cos x$ and $y = x^2 \cos x$.
a. Find the exact value of the area.
b. Use a trapezoidal approximation with n = 2 to approximate the area.
c. Use a midpoint approximation with n = 2 to approximate the area.
d. Find the Simpson approximation with n = 4.

12. Set up integrals to compute the volume of revolution generated by revolving the region above the x-axis and below $y = e^{-x^2}$ around the y-axis.
a. Use the shell method.
b. Use the disc method.
c. Evaluate one of these (note that both are improper integrals.)

13. A right circular cone has height h and base radius r and is buried so that its base is horizontal and its tip is at ground level. If the cone is filled to half its height with water, how much work is required to pump all the water out of the cone?



14. Find the x coordinate of the centroid of the portion of the unit circle in the first quadrant.

15. Solve the initial value problem:
$y(0) = \frac{\pi}{4}$ and $\frac{dy}{dx} = \sin^3 x \cos^2 y$
Hint $\sin^3 x = \sin x \sin^2 x = \sin x \left(1 - \cos^2 x\right)$

16. Use the integral test to determine whether the series converges or diverges: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

17. Find a Maclaurin series for $\ln\left(1 + x^2\right)$. *Hint*: integrate its derivative.
What is the radius of convergence?

18. Find the first two non-zero terms in the Taylor series for $g(x) = \int_1^{x^2} \arctan\left(t^2\right) dt$,
expanding about x = 1.

19. Use a binomial series to approximate $\sqrt[3]{9}$. How many terms are required to approximate to the nearest thousandth? *Hint*: $9 = 8\left(1 + \frac{1}{8}\right)$

Math 1B Final Exam Problems Solutions –

1. Find the area above the *x*-axis and below the curve $y=\dfrac{1-x^2}{1+x^2}$.

   ANS: The curve lies above the axis on (-1,1) so the area is

   $$\int_{-1}^{1}\frac{1-x^2}{1+x^2}dx=2\int_0^1\frac{1-x^2}{1+x^2}dx=2\int_0^1-1+\frac{2}{1+x^2}dx=-2x+4\arctan(x)\Big|_0^1=\pi-2$$

   Note that the usual partial fractions method requires the division as above first. If you feel compelled to use a trig-substitution, $x=\tan\theta$ yields $1+x^2=\sec^2\theta$ and $dx=\sec^2\theta\,d\theta$ so

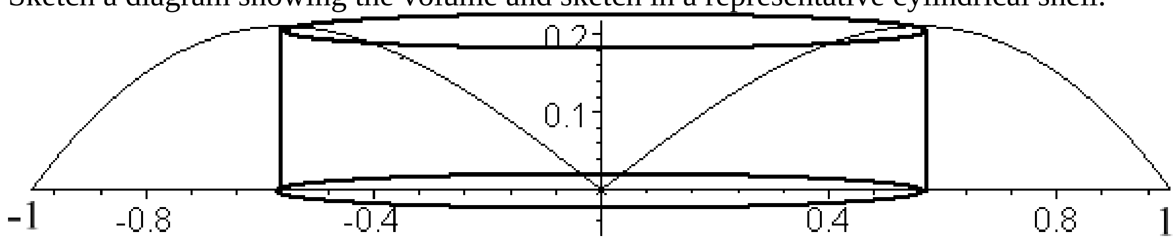
   $$2\int_0^1\frac{1-x^2}{1+x^2}dx=2\int_0^{\pi/4}\left(\frac{1-\tan^2\theta}{\sec^2\theta}\right)\sec^2\theta\,d\theta=2\int_0^{\pi/4}1-\tan^2\theta\,d\theta$$

   $$=2\int_0^{\pi/4}2-\sec^2\theta\,d\theta=4\theta-2\tan\theta\Big|_0^{\pi/4}=\pi-2$$

2. Suppose the area in the first quadrant bound by $y=-x+\sin\left(\dfrac{\pi x}{2}\right)$ is revolved around the *y*-axis.

   Assume all variables represent length in meters.

   a. Sketch a diagram showing the volume and sketch in a representative cylindrical shell.



   b. Set up an integral to compute the volume generated.

   $$2\pi\int_0^1 x\left(-x+\sin\left(\frac{\pi x}{2}\right)\right)dx$$

   c. Evaluate the integral to find the volume generated.

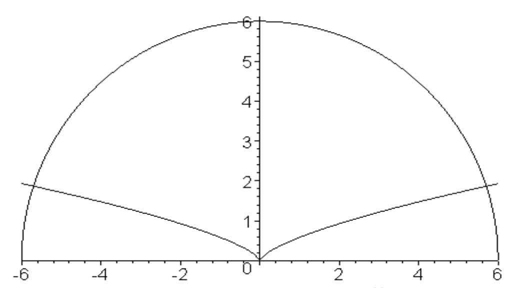
   $$2\pi\int_0^1 x\left(-x+\sin\left(\frac{\pi x}{2}\right)\right)dx=-\frac{2\pi}{3}x^3\Big|_0^1+2\pi\int_0^1 x\sin\left(\frac{\pi x}{2}\right)dx$$

   $$=-\frac{2\pi}{3}+2\pi\left(-\frac{2x}{\pi}\cos\left(\frac{\pi x}{2}\right)+\frac{4}{\pi^2}\sin\left(\frac{\pi x}{2}\right)\right)\Big|_0^1=\frac{8}{\pi}-\frac{2\pi}{3}=-\frac{2\left(\pi^2-12\right)}{3\pi}$$

3. Find the length of the curve $5y^3=x^2$ that lies inside the circle of radius 6 centered at the origin.

   The points of intersection are where $5y^3=x^2$ and $x^2+y^2=36$, that is where $5y^3+y^2=36$.

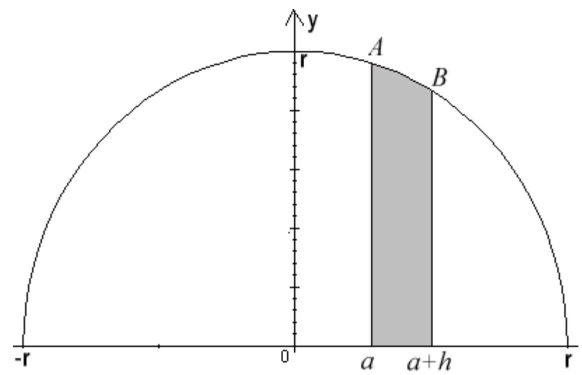
   As the graph below shows, the point of intersection is approximately $(5.70228,1.86656)$. Thus The length is approximately



$$2\int_0^{1.86656} \sqrt{1+(x')^2}\,dy = 2\int_0^{1.86656} \sqrt{1+\left(\frac{15y^2}{2x}\right)^2}\,dy = 2\int_0^{1.86656} \sqrt{1+\frac{225y^4}{4(5y^3)}}\,dy$$

$$= 2\int_0^{1.86656} \sqrt{\frac{45y+4}{4}}\,dy = \int_0^{1.86656} \sqrt{45y+4}\,dy = \frac{2}{135}(45y+4)^{3/2}\Big|_0^{1.86656} \approx 12.01103$$

4. If a spherical loaf of bread is cut into slices of equal width, each slice will have the same amount of crust. To verify this, suppose that a semicircle $y=\sqrt{r^2-x^2}$ is revolved about the x-axis to generate a sphere. Let AB be the arc of a semicircle that stands above an interval of length h on the x-axis. Show that the surface area swept out by AB does not depend on the location of the interval. That is, the length of AB in the diagram is independent of a.



ANS:

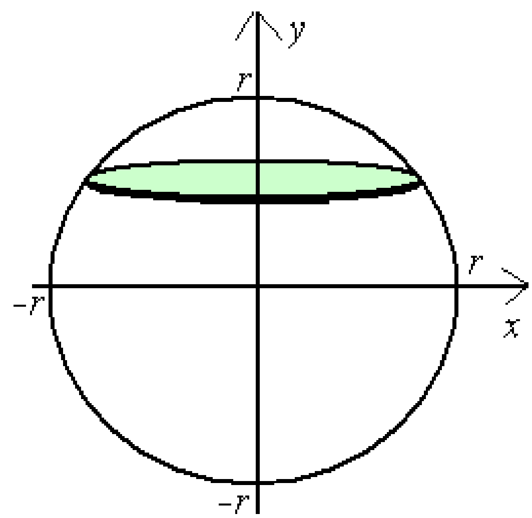
$$2\pi\int_a^{a+h} \sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx = 2\pi\int_a^{a+h} \sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}\,dx = 2\pi r\int_a^{a+h} dx = 2\pi r(a+h-a) = 2\pi rh$$

5. A spherical tank or radius r is filled with water. Find the minimum work required to empty the tank through a hole at its top. Be clear to show how your tank is oriented in a coordinate system.

$$W = \int dW = 9810\int_{-r}^{r} (r-y)\pi(r^2-y^2)\,dy$$

$$= 9810r\pi\int_{-r}^{r} (r^2-y^2)\,dy = 9810r\pi\left(r^2y-\frac{y^3}{3}\right)_{-r}^{r}$$

$$= 9810r\pi\left(\frac{4r^3}{3}\right) = 13080\pi r^4 \text{ Joules}$$



6. Show that the area of the region between the curve $y=\frac{1}{1+x^2}$ and the entire x-axis is the same as the area of the unit disk, $x^2+y^2\le 1$.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\int_0^{\infty} \frac{dx}{1+x^2} = 2\lim_{b\to\infty} \arctan(x) = \pi$$

7. Find the first three non-zero terms in the Maclaurin series for $f(x)=\cos^3 x$

   a. By cubing the Maclaurin series for $y=\cos x$

$$\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)^3 = \left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)^2\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right)$$

$$= \left(1-x^2+\frac{x^4}{6}-\frac{2x^6}{45}+\cdots\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots\right) \approx 1-\frac{3x^2}{2}+\frac{7x^4}{8}$$

b. By using the identity $\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$ and adding terms.

$$\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x \approx \frac{1}{4}\left(1 - \frac{9x^2}{2} + \frac{81x^4}{24}\right) + \frac{3}{4}\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) = 1 - \frac{12x^2}{8} + \frac{84x^4}{96}$$

8. Approximate the integral $\int_{-0.1}^{0.1} e^{-x^2}dx$ using

a. Simpson's rule with $n = 4$.

$$\int_{-0.1}^{0.1} e^{-x^2}dx = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$= \frac{1}{60}\left(e^{-0.01} + 4e^{-0.0025} + 2e^{0} + 4e^{-0.0025} + e^{-0.01}\right) \approx 0.19933541078$$

Or, using symmetry,

$$\int_{-0.1}^{0.1} e^{-x^2}dx = 2\int_{0}^{0.1} e^{-x^2}dx \approx \frac{1}{60}\left(1 + 4e^{-0.000625} + 2e^{-0.0025} + 4e^{-0.005625} + e^{-0.01}\right) \approx 0.199335333707$$

b. The Taylor polynomial of degree 6.

$$\int_{-0.1}^{0.1} e^{-x^2}dx \approx \int_{-0.1}^{0.1}\left(1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{x^6}{24}\right)dx = 2\left(x - \frac{x^3}{6} + \frac{x^5}{30} - \frac{x^7}{168}\right)_0^{0.1} \approx 0.199667332$$

9. Use a Maclaurin series to approximate $\frac{3}{e}$. How many terms are necessary to approximate this to the nearest 5×10^{-13}?

ANS: $\frac{3}{e} = 3e^{-1} = 3 - \frac{3}{2} + \frac{3}{6} - \frac{3}{24} + \frac{3}{120} - \frac{3}{720} + \cdots + \frac{3(-1)^n}{n!}$ is alternating so that $\frac{3}{16!} \approx 1.4\times10^{-13}$ suggests the approximation $3\sum_{n=0}^{15}\frac{(-1)^n}{n!} \approx 1.10363832351$ is accurate to the nearest 5×10^{-13}.

10. Use integration by parts to find the area below $y = e^{-x}\cos \pi x$ between $x = 0$ and $x = ½$.

$u = e^{-x}$	$dv = \cos \pi x dx$
$du = -e^{-x}dx$	$v = \frac{\sin \pi x}{\pi}$

$$I = \int e^{-x}\cos \pi x dx = \frac{e^{-x}\sin \pi x}{\pi} + \frac{1}{\pi}\int e^{-x}\sin \pi x dx$$

$u = e^{-x}$	$dv = \sin \pi x dx$
$du = -e^{-x}dx$	$v = \frac{-\cos \pi x}{\pi}$

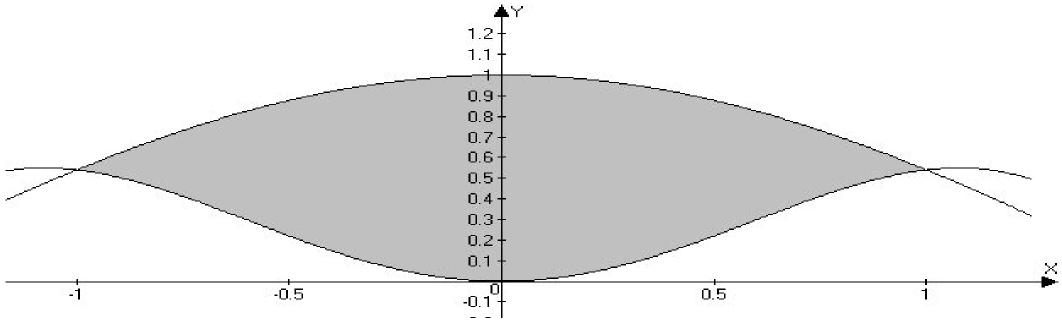
$$= \frac{e^{-x}\sin \pi x}{\pi} + \frac{1}{\pi}\left[\frac{-e^{-x}\cos \pi x}{\pi} - \frac{1}{\pi}I\right]$$

$$\Leftrightarrow \left(1 + \frac{1}{\pi^2}\right)I = \frac{e^{-x}}{\pi}\left(\sin \pi x - \frac{\cos \pi x}{\pi}\right) \Leftrightarrow I = \frac{\pi e^{-x}}{\pi^2 + 1}\left(\sin \pi x - \frac{\cos \pi x}{\pi}\right)$$

Thus the area is $\int_0^{1/2} e^{-x}\cos \pi x dx = \frac{\pi}{\pi^2 + 1}e^{-x}\left(\sin \pi x - \frac{\cos \pi x}{\pi}\right)\Bigg|_0^{1/2} = \boxed{\frac{\pi}{\left(\pi^2 + 1\right)}\left(\frac{1}{\sqrt{e}} + \frac{1}{\pi}\right)}$

11. Consider the area between the curves $y = \cos x$, $y = x^2\cos x$, $x = -1$ and $x = 1$.

a. Find the exact value of the area.

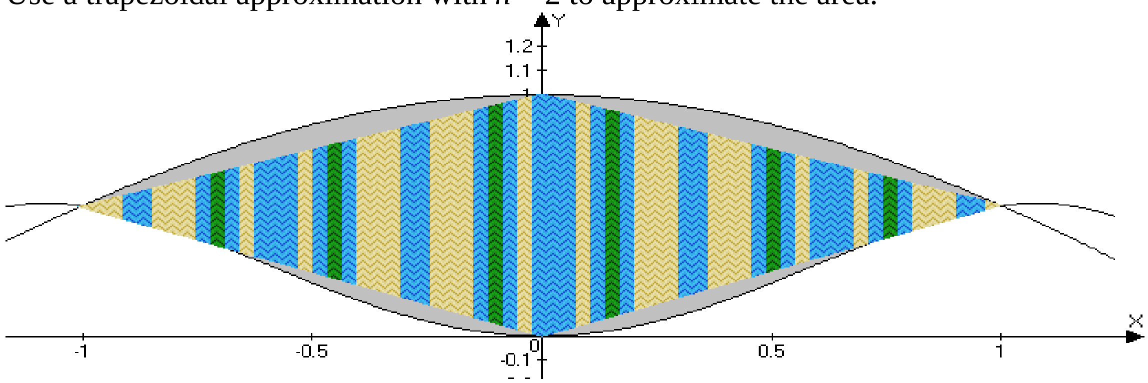


$$\int_{-1}^{1} \cos x - x^2 \cos x dx = \int_{-1}^{1} \left(1-x^2\right)\cos x dx \quad \boxed{\begin{array}{ll} u = 1-x^2 & dv = \cos x dx \\ du = -2x dx & v = \sin x \end{array}}$$

SOLN: $= \left(1-x^2\right)\sin x\Big|_{-1}^{1} + 2\int_{-1}^{1} x \sin x dx \quad \boxed{\begin{array}{ll} u = x & dv = \sin x dx \\ du = dx & v = -\cos x \end{array}}$

$$= 0 - 0 - 2x\cos x\Big|_{-1}^{1} + 2\int_{-1}^{1} \cos x dx = 4\left(\sin(1) - \cos(1)\right) \approx 1.20467$$

b. Use a trapezoidal approximation with $n = 2$ to approximate the area.



The trapezoidal area, as depicted above, is exactly 1 square unit: the sum of two triangular areas, each with base 1 (along the y axis) and height 1 (parallel to x-axis.)

c. Use a midpoint approximation with $n = 2$ to approximate the area.

SOLN: At the midpoints the height is $\cos(0.5)\left(1-0.5^2\right) = \frac{3}{4}\cos\left(\frac{1}{2}\right) \approx 0.65819$ so that the midpoint area is then $\Delta x\left(h_1 + h_2\right) = 1\left(0.65819 + 0.65819\right) \approx 1.32$

d. Find the Simpson approximation with $n = 4$.

SOLN: The Simpson approximation is a weighted average of the midpoint and trapezoid values: $\frac{2(1.32)+1}{3} \approx 1.21$

12. Set up integrals to compute the volume of revolution generated by revolving the region above the x-axis and below $y = e^{-x^2}$ around the y-axis.

a. Use the shell method.

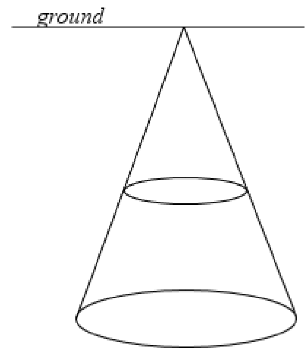
$$\int 2\pi r h dx = 2\pi \int_0^\infty x e^{-x^2} dx = \pi \int_0^\infty e^{-u} du = -\pi \lim_{b\to\infty} e^{-u}\Big|_0^b = -\pi \lim_{x\to\infty} e^{-b} - 1 = \pi$$

b. Use the disc method.

$$\int \pi r^2 dy = \pi \int_0^1 -\ln y dy = -\pi \lim_{b\to 0^+} y \ln y - y\Big|_b^1 = \pi + \pi \lim_{b\to 0^+} \frac{\ln y}{1/y} = \pi + \pi \lim_{b\to 0^+} \frac{1/y}{-1/y^2} = \pi$$

c. Evaluate one of these (note that both are improper integrals.)|
SOLN: Both are shown above.

13. A right circular cone has height h and base radius r and is buried so that its base is horizontal and its tip is at ground level. If the cone is filled to half its height with water, how much work is required to pump all the water out of the cone?



SOLN: Place the origin of the x axis at the tip of the cone and increasing down along the axis of the cone. By similar triangles the radius R of the cross-section at x satisfies $R/x = r/h$ so the volume is

$\int$ Force density $\times$ depth $\times$ element of volume $=$

$$9800\pi\int_{h/2}^{h} x\left(\frac{rx}{h}\right)^2 dx = \frac{9800\pi r^2}{h^2}\int_{h/2}^{h} x^3 dx$$
$$= \frac{9800\pi r^2}{h^2}\frac{x^4}{4}\bigg|_{h/2}^{h} = \boxed{\frac{18375\pi h^2 r^2}{8}}$$

14. Find the x coordinate of the centroid of the portion of the unit circle in the first quadrant.

SOLN: $\bar{x} = \dfrac{\int_0^1 x\sqrt{1-x^2}\,dx}{\pi/4} = \dfrac{4}{\pi}\int_0^{\pi/2} \sin t\cos^2 t\,dt = -\dfrac{4}{\pi}\int_1^0 u^2 du = \dfrac{4u^3}{3\pi}\bigg|_0^1 = \dfrac{4}{3\pi}$

15. Solve the initial value problem:

$y(0) = \dfrac{\pi}{4}$ and $\dfrac{dy}{dx} = \sin^3 x\cos^2 y$

SOLN:
$$\frac{dy}{dx} = \sin^3 x\cos^2 y \Leftrightarrow \int \sec^2 y\,dy = \int \sin^3 x\,dx \Leftrightarrow \tan y = \int \sin x - \sin x\cos^2 x\,dx$$
$$\Leftrightarrow \tan y = -\cos x - \frac{1}{3}\cos^3 x + c \Leftrightarrow y = \arctan\left(\frac{7-\cos^3 x}{3} - \cos x\right)$$

16. Use the integral test to determine whether the series converges or diverges: $\sum_{n=1}^{\infty}\dfrac{1}{\sqrt{n^3}}$

SOLN The terms are positive and decreasing and $\int_1^{\infty} x^{-3/2}dx = -2\lim_{b\to\infty} x^{-1/2}\Big|_1^b = 2$ is finite, so the series is convergent.

17. Find a Maclaurin series for $\ln(1+x^2)$. *Hint*: integrate its derivative.

What is the radius of convergence?

SOLN:
$$\ln(1+x^2) = \int \frac{d}{dx}\ln(1+x^2)\,dx = \int \frac{2x}{1+x^2}dx = 2\int x\sum_{n=0}^{\infty}(-x^2)^n dx = 2\sum_{n=0}^{\infty}\int (-1)^n x^{2n+1}dx$$
$$= 2\sum_{n=0}^{\infty}\int (-1)^n x^{2n+1}dx = 2\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{2n+2} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{n+1}$$

The radius of convergence is 1.

18. Find the first two non-zero terms in the Taylor series for $g(x) = \int_1^{x^2} \arctan(t^2)\,dt$, expanding about $x = 1$.

SOLN: $c_0 = g(1) = \int_1^1 \arctan(t^2)\,dt = 0$, $c_1 = 2x\arctan(x^2)\Big|_{x=1} = \frac{\pi}{2}$ and

$c_2 = 2\arctan(x^2) + \frac{4x^2}{1+x^2}\Big|_{x=1} = \frac{\pi}{2} + 2$ so $g(x) = \int_1^{x^2} \arctan(t^2)\,dt \approx \boxed{\frac{\pi}{2}(x-1) + \left(\frac{\pi}{4}+1\right)(x-1)^2}$

19. Use a binomial series to approximate $\sqrt[3]{9}$. How many terms are required to approximate to the nearest thousandth?

$$\sqrt[3]{9} = \left[8\left(1+\frac{1}{8}\right)\right]^{1/3} = 2\left(1+\frac{1}{8}\right)^{1/3} = 2\sum_{n=0}^{\infty}\binom{1/3}{n}\left(\frac{1}{8}\right)^n \approx 2 + \frac{2}{3\cdot 8} - \frac{2^2}{2\cdot 3^2 8^2} + \frac{2^2 5}{3!\cdot 3^3 8^3}.$$

$$= 2 + \frac{1}{12} - \frac{1}{288} + \frac{5}{20736}$$

Evidently, this series is alternating so, since the 4th term is less than half of a thousandth, only 3 terms are needed to approximate to the nearest thousandth: $\sqrt[3]{9} \approx 2 + \frac{1}{12} - \frac{1}{288} = \frac{599}{288} \approx 2.080$